



Remarks on the Huygens absorbing boundary conditions for electromagnetics

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ABSTRACT

The Huygens absorbing boundary conditions (ABCs) are promising new implementations of operator ABCs. They have certain advantageous features which are lacked in other operator ABCs. Under certain conditions and with the Huygens ABCs, the transmitted wave depends solely upon the second derivative with respect to time or upon the double integral of the incident wave. For such cases and for problems with a Dirichlet boundary condition, the overall reflection is not unique. Two new examples of the Huygens ABCs are given for such cases. For each example and with a FDTD scheme the newly derived reflection is less than that which has been studied by Bérenger [J.-P. Bérenger, On the Huygens absorbing boundary conditions for electromagnetics, J. Comput. Phys. 226 (2007) 354–378].

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1. Introduction

Absorbing boundary conditions (ABCs for short) are needed for the solutions of electromagnetic problems with a numerical method, e.g., finite-difference time-domain (FDTD) method [1] on a bounded domain. Historically, many researchers, e.g., [2–13] have contributed to develop absorbing (radiating) or non-reflecting boundary conditions for wave propagation problems. Perhaps the most widely known ones are the local boundary conditions [4] by Engquist and Majda, and its extension [6]. The absorbing boundary conditions [4,6] have been generalized by Higdon with the aid of dispersion relation of the difference approximation of the wave equation [8,9]. In this aspect one should mention the work by Trefethen and Halpern [14]. Comparative studies of several absorbing (radiating) boundary conditions have been carried out by Blaschak and Kriegsmann [15], and by Railton and Daniel [16].

In the relatively recent paper [13], Bérenger has proposed the so-called Huygens absorbing boundary conditions for the numerical solutions of electromagnetic wave problems. The idea [13] is based upon the equivalence theorem (principle) of electromagnetics [17,18]. The re-radiating boundary condition (rRBC) [11,12] and the multiple absorbing surfaces (MAS) [10] are respectively special cases of the Huygens ABCs [13]. Bérenger has shown rigorously that the Huygens ABCs are theoretically equivalent to previously known absorbing boundary conditions which are based on operators, e.g., the one by Higdon [8,9]. Nevertheless, though not brand new, the Huygens ABCs have certain properties that are valuable to numerical solutions of electromagnetic problems, e.g., free of stability issue for higher-order boundary conditions, and special design tailored to absorb both traveling and evanescent waves [13], etc. Thus, the Huygens ABCs are novel implementations of boundary conditions which are based on operators, and for certain problems could “challenge” [13] the one that is based on the concept of perfectly matched layer [19,20].

With the so-called elementary operator P_e and with first-order expansion of the estimate of the incident field \tilde{U} , Bérenger has shown that the Huygens ABCs have the non-dispersive property, i.e., the reflected wave is just a copy of the incident wave [13]. Further, the *unique* formula of the overall reflection has been derived [13]. It has been pointed out [13] that under certain conditions and with the spatial and temporal shift operators, the first-order terms in the expansion of incident

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field may automatically drop out. For such cases, the second derivatives are involved in the transmitted wave. Moreover, depending upon the specific form with the shift operators, the overall reflection may be different, i.e., not *unique* when the expansion of \tilde{U} is free of first-order terms. In this study we are concerned with the non-uniqueness of the overall reflection for such cases.

2. Operators free of first-order terms

The linear operator used for the approximation of the incident field takes the form [13]

$$\mathcal{L} := \sum_{n=1}^N c_n \mathcal{K}(-\delta_{x_n}) \mathcal{Z}(-\delta_{t_n}), \quad (1)$$

where \mathcal{K} and \mathcal{Z} are respectively the spatial and temporal shift operators, defined as

$$\mathcal{K}(-\delta_x)u(x, t) := u(x - \delta_x, t), \quad \mathcal{Z}(-\delta_t)u(x, t) := u(x, t - \delta_t). \quad (2)$$

The expansion for the estimate of the incident field can be expressed as

$$\tilde{U}(x_c, t) = \mathcal{L}u_a(x_c, t), \quad (3)$$

where $u_a(x_c, t)$ is the actual field [13]. For a certain form of (1), the first-order terms will automatically cancel in the expansion (3). It is easy to show that such a cancellation happens if the *total* number of shifts with a positive sign is the same as that with a negative sign, and this has to hold for both spatial and temporal variables. The example in [13]

$$\mathcal{L}_B := \mathcal{K}(-\delta_x) + \mathcal{K}(-\delta_x)\mathcal{Z}(-\delta_t) - \mathcal{K}(-2\delta_x)\mathcal{Z}(-\delta_t) \quad (4)$$

satisfies such conditions. For the estimate with the operator (4) Béranger has derived the overall reflection [13]. For the completeness of this study, we record the result as follows

$$r_B = -\frac{\delta_x - c\delta_t}{\delta_x + c\delta_t}, \quad (5)$$

where c is the speed of light. In this study we give two more examples which satisfy the above conditions. The goal of these examples is to show that under such conditions (i) the overall reflection is *not unique*, and (ii) for certain cases, the overall reflection may be *smaller* than the one in (5).

3. The first case

We first consider the case with the operator defined as

$$\mathcal{L}_{c_1} := 2\mathcal{K}(-\delta_x)\mathcal{Z}(-\delta_t) - \mathcal{K}(-2\delta_x)\mathcal{Z}(-2\delta_t). \quad (6)$$

For the wave that travels to the right, the estimate of the incident wave is

$$\tilde{U}(x_c, t) = 2u_{i+}(x_c - \delta_x, t - \delta_t) - u_{i+}(x_c - 2\delta_x, t - 2\delta_t). \quad (7)$$

For the two terms on the right-hand side of (7), we use the Taylor expansion about the point (x_c, t) up to the second order. For the first term, we have

$$2u_{i+}(x_c - \delta_x, t - \delta_t) = 2 \left[u_{i+} - \frac{\partial u_{i+}}{\partial x} \delta_x - \frac{\partial u_{i+}}{\partial t} \delta_t + \frac{1}{2} \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2 + \frac{1}{2} \frac{\partial^2 u_{i+}}{\partial t^2} \delta_t^2 + \frac{\partial^2 u_{i+}}{\partial x \partial t} \delta_x \delta_t \right]. \quad (8)$$

For the second term, one ends up with

$$u_{i+}(x_c - 2\delta_x, t - 2\delta_t) = u_{i+} - 2 \frac{\partial u_{i+}}{\partial x} \delta_x - 2 \frac{\partial u_{i+}}{\partial t} \delta_t + 2 \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2 + 2 \frac{\partial^2 u_{i+}}{\partial t^2} \delta_t^2 + 4 \frac{\partial^2 u_{i+}}{\partial x \partial t} \delta_x \delta_t. \quad (9)$$

Substituting (8) and (9) into (7), and after canceling first-order like terms, we have

$$\tilde{U}(x_c, t) = u_{i+}(x_c, t) - \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2 - \frac{\partial^2 u_{i+}}{\partial t^2} \delta_t^2 - 2 \frac{\partial^2 u_{i+}}{\partial x \partial t} \delta_x \delta_t. \quad (10)$$

The transmitted field is given by [13]

$$U_{t+}(x_{c+}, t) = u_{i+}(x_c, t) - \tilde{U}(x_c, t). \quad (11)$$

Plugging (10) into (11), we yield

$$U_{t+}(x_{c+}, t) = \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2 + \frac{\partial^2 u_{i+}}{\partial t^2} \delta_t^2 + 2 \frac{\partial^2 u_{i+}}{\partial x \partial t} \delta_x \delta_t. \quad (12)$$

For the incident wave that propagates to the right, we assume that the wave function has the following form

$$u_{i+}(x, t) = f\left(t - \frac{x}{c}\right), \quad (13)$$

where $f(\bullet)$ is a differentiable function up to the second order. In view of (13), the following holds in terms of the partial derivatives of the second order

$$\frac{\partial^2 u_{i+}}{\partial x^2} = \frac{f''}{c^2}, \quad \frac{\partial^2 u_{i+}}{\partial t^2} = f'', \quad \frac{\partial^2 u_{i+}}{\partial x \partial t} = -\frac{f''}{c}. \quad (14)$$

Substituting the above partial derivatives into (12), we have

$$U_{t+}(x_{c+}, t) = f'' \left(\frac{\delta_x^2}{c^2} + \delta_t^2 - 2\frac{\delta_x \delta_t}{c} \right) = -\frac{\partial^2 u_{i+}}{\partial t^2} \frac{\delta_x}{c} \left(2\delta_t - c\frac{\delta_t^2}{\delta_x} - \frac{\delta_x}{c} \right). \quad (15)$$

Comparing the expression (A.5) in [13] of the transmitted wave for the case with the operator in (4), we see that the scaling factor in (15) is different from that in (A.5) though qualitatively the characteristic of the transmitted wave is still the same—proportional to the second partial derivative of the incident field with respect to the temporal variable.

We now consider the case when the wave travels to the left. With the same expansion up to the second order, one can show that the estimate of the incident wave has the form

$$\tilde{U}(x_c, t) = U_{t-}(x_c, t) - \frac{\partial^2 U_{t-}}{\partial x^2} \delta_x^2 - \frac{\partial^2 U_{t-}}{\partial t^2} \delta_t^2 - 2\frac{\partial^2 U_{t-}}{\partial x \partial t} \delta_x \delta_t. \quad (16)$$

For the wave traveling to the left, the following holds for the transmitted wave [13]

$$U_{t-}(x_{c-}, t) = u_{i-}(x_c, t) + \tilde{U}(x_c, t). \quad (17)$$

Plugging the expression of $\tilde{U}(x_c, t)$ into the above equation, one gets

$$u_{i-}(x_c, t) = \frac{\partial^2 U_{t-}}{\partial x^2} \delta_x^2 + \frac{\partial^2 U_{t-}}{\partial t^2} \delta_t^2 + 2\frac{\partial^2 U_{t-}}{\partial x \partial t} \delta_x \delta_t. \quad (18)$$

For the transmitted wave that goes to the left, we assume that the wave function has the form

$$U_{t-}(x, t) = g\left(t + \frac{x}{c}\right), \quad (19)$$

where $g(\bullet)$ is again a differentiable function up to the second order. In view of (19), the following holds in terms of the partial derivatives of the second order

$$\frac{\partial^2 U_{t-}}{\partial x^2} = \frac{g''}{c^2}, \quad \frac{\partial^2 U_{t-}}{\partial t^2} = g'', \quad \frac{\partial^2 U_{t-}}{\partial x \partial t} = \frac{g''}{c}. \quad (20)$$

Substituting the above expressions into (18) yields

$$u_{i-}(x_c, t) = g'' \left(\frac{\delta_x^2}{c^2} + \delta_t^2 + 2\frac{\delta_x \delta_t}{c} \right) = \frac{\partial^2 U_{t-}}{\partial t^2} \frac{\delta_x}{c} \left(2\delta_t + c\frac{\delta_t^2}{\delta_x} + \frac{\delta_x}{c} \right). \quad (21)$$

So the left-going transmitted wave can be inferred from (21)

$$U_{t-}(x_{c-}, t) = \frac{1}{\frac{\delta_x}{c} \left(2\delta_t + c\frac{\delta_t^2}{\delta_x} + \frac{\delta_x}{c} \right)} \int^t \left[\int^\tau u_{i-}(x_c, \sigma) d\sigma \right] d\tau. \quad (22)$$

If we compare the scaling coefficient in (22) with the one (A.8) in [13] which is for the case with the operator (4), we find that the two coefficients are different.

We may combine the results in (15), (22) by considering the following problem. An incident wave propagates to the right, and is reflected back from the wall with the Dirichlet condition $U = 0$, which corresponds to the perfect electric condition (PEC) or perfect magnetic condition (PMC) in electromagnetics [13]. It is easy to show that for such a problem the overall reflection is given by

$$r_{c1} = -\frac{\delta_x + c^2 \frac{\delta_t^2}{\delta_x} - 2c\delta_t}{\delta_x + c^2 \frac{\delta_t^2}{\delta_x} + 2c\delta_t}. \quad (23)$$

The reflection coefficient r_{c1} in (23) is different from r_B in (5). It is easy to show that under the condition $c\delta_t < \delta_x$, the newly derived reflection coefficient r_{c1} is less than r_B , i.e.,

$$r_{c1} < r_B \quad \text{if } c\delta_t < \delta_x. \quad (24)$$

The condition is definitely satisfied if $\delta_x = \Delta_x$, and $\delta_t = \Delta_t$, where Δ_x and Δ_t are the step sizes of the spatial and temporal variables in a FDTD scheme, respectively.

4. The second case

Now we consider the case with the operator defined as

$$\mathcal{L}_{c_2} := \mathcal{K}(-\delta_x) + \mathcal{K}(-\delta_x)\mathcal{Z}(-2\delta_t) - \mathcal{K}(-2\delta_x)\mathcal{Z}(-2\delta_t). \quad (25)$$

For the wave which propagates to the right, the estimate of the incident wave is given by

$$\tilde{U}(x_c, t) = u_{i+}(x_c - \delta_x, t) + u_{i+}(x_c - \delta_x, t - 2\delta_t) - u_{i+}(x_c - 2\delta_x, t - 2\delta_t). \quad (26)$$

For the first two terms on the right-hand side of (26), we use the Taylor expansion about the point (x_c, t) up to the second order. For the first term, we have

$$u_{i+}(x_c - \delta_x, t) = u_{i+} - \frac{\partial u_{i+}}{\partial x} \delta_x + \frac{1}{2} \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2. \quad (27)$$

And for the second term, we have

$$u_{i+}(x_c - \delta_x, t - 2\delta_t) = u_{i+} - \frac{\partial u_{i+}}{\partial x} \delta_x - 2 \frac{\partial u_{i+}}{\partial t} \delta_t + \frac{1}{2} \frac{\partial^2 u_{i+}}{\partial x^2} \delta_x^2 + 2 \frac{\partial^2 u_{i+}}{\partial t^2} \delta_t^2 + 2 \frac{\partial^2 u_{i+}}{\partial x \partial t} \delta_x \delta_t. \quad (28)$$

The Taylor expansion of the third term on the right-hand side of (26) is given by (9). With the help of (9), (27) and (28) and following the same procedure as in the first case, we can show that for the wave traveling to the right, the transmitted wave is given by

$$U_{t+}(x_{c+}, t) = -\frac{\partial^2 u_{i+}}{\partial t^2} \frac{\delta_x}{c} \left(2\delta_t - \frac{\delta_x}{c} \right), \quad (29)$$

which is different from the one (A.5) in [13] for the case with the operator \mathcal{L}_B in (4). Similarly for the wave that propagates to the left, the transmitted wave takes the form

$$U_{t-}(x_{c-}, t) = \frac{1}{\frac{\delta_x}{c} (2\delta_t + \frac{\delta_x}{c})} \int^t \left[\int^\tau u_{i-}(x_c, \sigma) d\sigma \right] d\tau, \quad (30)$$

which is again different from the one (A.8) in [13] for the case with the operator \mathcal{L}_B in (4). In view of the results in (29) and (30) the overall reflection for the second case is shown as

$$r_{c_2} = -\frac{\delta_x - 2c\delta_t}{\delta_x + 2c\delta_t}, \quad (31)$$

which differs from r_B in (5), and also from r_{c_1} in (23). It is easy to see that the reflection coefficient r_{c_2} in the second case is always lower than r_B , the one given by Bérenger [13], i.e.,

$$r_{c_2} < r_B. \quad (32)$$

5. Conclusion

The Huygens ABCs initiated by Bérenger in [13] have unified a couple of previous studies [10–12] on absorbing boundary conditions for computational electromagnetics. With a certain specification [13], the Huygens ABCs also embrace the operator ABCs which have been developed by Higdon [8,9]. The Huygens ABCs have some unique features that distinguish them from other operator ABCs. Under certain conditions and with the Huygens ABCs the transmitted wave can be expressed in terms of either only the second partial derivative of the incident wave or only its double integral, i.e., free from first partial derivatives and from a single integral. For such cases and for the problem with a Dirichlet boundary condition we have shown that the overall reflection coefficient is not unique, and it depends on the specific form with the shift operators. Two concrete examples are given for the purpose of illustration. In the first case, under normal conditions in a FDTD scheme the reflection is lower than the one [13] studied by Bérenger. And in the second case, the reflection is always smaller than that in [13].

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